

# Generalized Short Pulse Equation for Propagation of Few-Cycle Pulses in Metamaterials

Monika E. Pietrzyk and Igor V. Kanattšikov

School of Physics and Astronomy, University of St Andrews  
North Haugh, KY16 9SS St Andrews, UK  
mp212@st-andrews.ac.uk

**Abstract:** We show that propagation of ultrashort (few-cycle) pulses in nonlinear Drude metamaterials with both electric and magnetic Kerr nonlinearities is described by coupled generalized Short Pulse Equations. The resulting system of equations generalizes to the case of metamaterials both the Short Pulse Equation and its vector generalizations.

Keywords: metamaterials, optical fibers, ultra-short pulses, few-cycle pulses, short pulse equation.

PACS: 78.67.Pt, 42.69.Tg, 42.65.Re, 05.45.Yv, 75.78.Jp.

Metamaterials are artificial structures that display properties beyond those available in naturally occurring materials. The most notable are the negative refraction index materials with simultaneously negative electric and magnetic dispersive responses, which affect substantially the conventional optics and its applications. Metamaterials with the negative refraction index have a number of extraordinary properties, such as the reversed Snell refraction, reversed Doppler effect, reversed radiation tension, negative Cerenkov radiation, reversed Goos-Hänchen shift, etc. Such materials can be used in various devices, such as compact cavities, superlenses, subwavelength waveguides and antennas, electromagnetic cloaking devices, tunable mirrors, isolators, phase compensators, and many others. Metamaterials can be also used to study ideas developed to describe physics in curved space-times and to model virtually any space-time metric of General Relativity.

A typical metamaterial with the negative refraction index is composed of a combination of a regular array of electrically small resonant particles, referred to as split-ring resonators, and a regular array of conducting wires responsible, respectively, for the negative electric permittivity and negative magnetic permeability. The size and spacing of these elements is supposed to be much smaller than the wavelength of the propagating optical field, so that the metamaterial can be considered as a homogeneous medium.

Different models have been used to describe propagation of short and ultrashort optical pulses in metamaterials. For metamaterials without nonlinear magnetization a generalized nonlinear Schrödinger equation (NSE) has been derived [1, 2, 3, 4, 5]. For metamaterials with nonlinear magnetizations a single-component NSE for the electric field has been obtained in [6]. For the sufficiently long temporally optical pulses a system of coupled NSE can be used [7, 8, 9, 10].

However, for ultrashort few-cycle pulses the envelope approximation is not valid and the NSE cannot be applied. For metamaterials without the nonlinear magnetization a single-component generalized SPE for the electric field has been obtained in [11, 12]. When the nonlinear magnetization is present the equations for the electric and magnetic field cannot be uncoupled [13]. Here we show that the propagation of few-cycle pulses in metamaterials with electric and magnetic Kerr-type nonlinear response can be described by a coupled system of Short Pulse Equations for the electric and magnetic field.

It has been shown that a composite metamaterial with negative refraction index can develop a nonlinear macroscopic magnetic response. This means that, although the host medium has a negligible

magnetic nonlinearity, the periodic inclusions of the metamaterial can produce an effective magnetic nonlinear response.

Thus, let us consider a metamaterial with nonlinear magnetization. Let us start from the Maxwell equations for an optical field propagation along the z-direction.

$$\partial_z E = -\partial_t B - \partial_t M_{nl},$$

$$-\partial_z H = \partial_t D + \partial_t P_{nl},$$

$$\partial_x D = 0,$$

$$\partial_y B = 0,$$

where it is assumed that the electric and magnetic fields are linearly polarized:

$$\mathbf{E} = (E, 0, 0), \quad \mathbf{H} = (0, H, 0).$$

The dielectric and magnetic response of a nonlinear material is characterized by electric displacement field  $\tilde{D}(\omega) = \epsilon(\omega)\tilde{E}(\omega)$ , magnetic induction  $\tilde{B}(\omega) = \mu(\omega)\tilde{H}(\omega)$ , nonlinear polarization  $P_{nl} = \epsilon_{nl}E$ , and nonlinear magnetization  $M_{nl} = \mu_{nl}H$ .

Let us assume that both the electric and the magnetic nonlinearities are of Kerr type,  $\epsilon_{nl} = \chi_e E^2$  and  $\mu_{nl} = \chi_m H^2$ . Substituting the material equations into the Maxwell equations we obtain in the frequency domain:

$$\partial_z \tilde{E} = -i\omega\mu(\omega)\tilde{H} - i\omega\chi_m \tilde{H}^3, \quad (1)$$

$$\partial_z \tilde{H} = -i\omega\epsilon(\omega)\tilde{E} - i\omega\chi_e \tilde{E}^3. \quad (2)$$

Acting with  $\partial_z$  on equation (1) and using equation (2) we get:

$$\partial_{zz}\tilde{E} = -\omega^2 \left( \mu(\omega)\epsilon(\omega) + \mu(\omega)\chi_e \tilde{E}^2 + 3\epsilon(\omega)\chi_m \tilde{H}^2 + 3\chi_e\chi_m \tilde{E}^2 \tilde{H}^2 \right) \tilde{E}. \quad (3)$$

Similarly, acting with  $\partial_z$  on equation (2) and using equation (1) we obtain:

$$\partial_{zz}\tilde{H} = -\omega^2 \left( \epsilon(\omega)\mu(\omega) + \epsilon(\omega)\chi_m \tilde{H}^2 + 3\mu(\omega)\chi_e \tilde{E}^2 + 3\chi_e\chi_m \tilde{E}^2 \tilde{H}^2 \right) \tilde{H}. \quad (4)$$

Now, let us assume that the dispersive properties of the metamaterial are given by the lossless Drude model:

$$\epsilon(\omega) = \epsilon_0(1 - \omega_e^2/\omega^2) \quad \text{and} \quad \mu(\omega) = \mu_0(1 - \omega_m^2/\omega^2),$$

where  $\omega_e$  and  $\omega_m$  are the electric and magnetic plasma frequencies, respectively. Then

$$\epsilon(\omega)\mu(\omega) \approx \epsilon_0\mu_0 \left( 1 - \frac{\omega_e^2}{\omega^2} - \frac{\omega_m^2}{\omega^2} + \frac{\omega_e^2\omega_m^2}{\omega^4} \right).$$

Neglecting the term proportional to  $\omega^{-4}$  and using  $\epsilon_0\mu_0 = 1/c^2$ , where  $c$  is the velocity of light in vacuum, we obtain:

$$\epsilon(\omega)\mu(\omega) \approx \epsilon_0\mu_0(1 - \omega_e^2/\omega^2 - \omega_m^2/\omega^2).$$

Substituting these formulas into equations (3) and (4), applying the Fourier transform and neglecting higher order nonlinear terms proportional to  $E^2H^2$  we obtain:

$$\partial_{zz}E = \frac{\partial_{tt}E}{c^2} + \frac{\omega_e^2 + \omega_m^2}{c^2}E + \mu_0\chi_e\partial_{tt}(E^3) + \mu_0\omega_m^2\chi_e E^3 + 3\epsilon_0\chi_m\partial_{tt}(H^2E) + 3\epsilon_0\omega_e^2\chi_m H^2E \quad (5)$$

and

$$\partial_{zz}H = \frac{\partial_{tt}H}{c^2} + \frac{\omega_e^2 + \omega_m^2}{c^2}H + \epsilon_0\chi_m\partial_{tt}(H^3) + \epsilon_0\omega_e^2\chi_mH^3 + 3\mu_0\chi_e\partial_{tt}(E^2H) + 3\mu_0\omega_m^2\chi_eE^2H. \quad (6)$$

Introducing new variables:  $\tau = t - z/c$  and  $\zeta = z$  for which  $\partial_{zz} = 1/c^2\partial_{\tau\tau} + 2/c\partial_{\zeta\tau} + \partial_{\zeta\zeta}$  and  $\partial_{tt} = \partial_{\tau\tau}$  in equations (5) and (6) and making use of the paraxial approximation:  $\partial_{\zeta\zeta}E = \partial_{\zeta\zeta}H = 0$ , we obtain:

$$\partial_{\zeta\tau}E = \frac{\omega_e^2 + \omega_m^2}{2c}E + \frac{1}{2}\mu_0\chi_e c \left( \partial_{\tau\tau} + \omega_m^2 \right) E^3 + \frac{3}{2}\epsilon_0\chi_m c (\partial_{\tau\tau} + \omega_e^2)H^2E \quad (7)$$

and

$$\partial_{\zeta\tau}H = \frac{\omega_e^2 + \omega_m^2}{2c}H + \frac{1}{2}\epsilon_0\chi_m c \left( \partial_{\tau\tau} + \omega_e^2 \right) H^3 + \frac{3}{2}\mu_0\chi_e c (\partial_{\tau\tau} + \omega_m^2)E^2H. \quad (8)$$

Thus, we have obtained the set of two coupled generalized Short Pulse Equations. In the limit  $\omega_m \rightarrow 0$  and  $\chi_m \rightarrow 0$  equation (7) reduces to the Short Pulse Equation derived by Schäfer and Wayne [16]

$$\partial_{\zeta\tau}E = \frac{\omega_e^2}{2c}E + \frac{1}{2}\mu_0\chi_e c \partial_{\tau\tau}(E^3), \quad (9)$$

which later was shown to be an integrable system [15,16]. With certain combination of the parameters of the metamaterial we can also obtain from equations (9) and (8) different integrable vector generalizations of the short pulse equation obtained by us [17] and other authors [18,19].

In conclusion, a consideration of the propagation of ultra-short few-cycle polarized optical pulses in the Drude metamaterial optical fibers with electric and magnetic Kerr nonlinearity leads to the coupled set of equations which generalizes the Short Pulse Equation. It allows to describe the ultra-short and spectrally broad optical pulses beyond the slow varying envelope approximation. It may open a possibility of studying a new class of optical phenomena in metamaterials when the spectral range of the optical field overlaps the regions with different signs of optical indices of the metamaterial.

## REFERENCES.

- [1] M. Scalora, et al., Phys. Rev. Lett. 95, 013902 (2005),
- [2] S. Wen, et al., Opt. Express 14, 1568 (2006),
- [3] S. Wen, et al., Phys. Rev. E 73, 036617 (2006),
- [4] Hu Yong-Hua, et al., Chin. Phys. 15, 2970 (2006),
- [5] A. Kumar Mishra and A. Kumar, J. Mod. Opt. 59, 1599 (2012),
- [6] J. Zhang, et al., Phys Rev. A 81, 023829 (2010),
- [7] I. Kurakis and P. K. Shukla, Phys. Rev. E 72, 016626 (2005),
- [8] N. Lazarides and G. P. Tsironis, Phys. Rev. E 71, 036614 (2005)
- [9] S. Wen, et al., Phys. Rev. A 75, 033815 (2007),
- [10] N. L. Tsitses, et al., , Phys. Rev. E 79, 037601 (2009),

- [11] N. L. Tsitsas, et al., Phys. Lett. A 374, 1384 (2010),
- [12] Y. Shen, et al., Phys. Rev. A 86, 023841 (2012),
- [13] P. Kinsler, Phys. Rev. A 81, 023808 (2010),
- [14] T. Schäfer and C.E. Wayne, Physica D 196, 90 (2004),
- [15] A. Sakovich and S. Sakovich, J. Phys. Soc. Japan 74, 239 (2005),
- [16] J. C. Brunelli, J. Math. Phys. 46, 123507 (2005).
- [17] M. E. Pietrzyk, I. Kanattikov, et al., J. Nonl. Math. Phys. 15, 162 (2008).
- [18] J.C. Brunelli, S. Sakovich, Phys. Lett. A 377, 80 (2012), J. Math. Phys. 54, 012701 (2013).
- [19] Y. Matsuno, J. Math. Phys. 52, 123702 (2011).